Symmetries and similarity solutions: An application to fluid mechanics

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Abstract

The free convective boundary–layer problem due to the motion of an elastic surface into an electrically conducting fluid is studied with group–theoretical methods. The symmetry groups admitted by the corresponding boundary value problem are obtained. Particular attention is paid on the group of scaling which provides the similarity solution of the problem. Also, the admissible form of the data, in order to be conformed to the obtained symmetries, is provided. Finally, with the use of the entailed similarity solution the problem is transformed to a boundary value problem of ODEs and is solved numerically.

Keywords: Symmetry groups; Similarity solutions; Infinitesimal generator; Laminar boundary-layer flow

1 Introduction

In this paper, we apply the so–called symmetry methods for a particular problem of fluid mechanics. The main advantage of such methods is that they can successfully be applied to non–linear differential equations. The symmetries of a differential equation are those continuous groups of transformations under which the differential equation remains invariant, that is, a symmetry group maps any solution to another solution. The interesting point is that, having obtained the symmetries of a specific problem, one can proceed further to find out the group–invariant solutions, which, in the case of the scaling group of transformations, are nothing but the well–known similarity solutions. The similarity solutions are quite popular because they result in the reduction of the independent variables of the problem. In our case, the problem under investigation is two–dimensional. Hence, any similarity solution will transform the system of PDEs to a system of ODEs.

To obtain a symmetry of a differential equation is equivalent to the determination of the infinitesimal generator of the transformation group associated with this symmetry. In [1, 2, 3, 4], one can find the general theory of Lie groups as well as the implied methods for determining the infinitesimal generator components. An alternative way being based on exterior calculus for determining of the infinitesimal generator can be found in the book of Edelen [5]. It is worth to note that there is an extensive literature where the methods arising from exterior calculus are used to attack symmetry problems of continuum mechanics [6, 7, 8, 9, 10, 11, 12].

Most of the researchers in the field of fluid mechanics usually try to obtain the similarity solutions by introducing a general similarity transformation with unknown parameters into the differential equation obtaining in this way an algebraic system. Then, the solution of this system, if exists, determines the values of the unknown parameters. In our opinion, it is better to attack any problem of similarity solutions from the outset, i.e, to find out the full list of the symmetries of the problem and then to study which of them are appropriate to provide group–invariant (or more specifically similarity solutions).

We apply this procedure to a boundary layer problem which arises from the motion of an elastic surface into an electrically conducting, incompressible viscous fluid. Particular variants of this problem have been studied by a numerous of researchers since 1961. We mention here the work of [13, 14, 15, 16, 17, 18, 19, 20]. It is remarkable that all of them have used the above described heuristic method to obtain the similarity transformations and the associated similarity solutions of the problem. That is, assuming particular boundary conditions and considering a particular form of the magnetic field[2], they try to fit a similarity solution in these data.

We set the problem on another base. First, we do not guess any kind of probable symmetry. The question of any possible symmetry for the system of PDEs is examined generally. In the same spirit, we do not make any assumption about the data of the problem. We consider the most general form for the boundary conditions and the magnetic field function involved in the system. Both the form of the functions on the boundaries and the form of magneting field arise as a consequence of the requirement to respect the obtained symmetries.

Next, having established the admissible symmetries of the boundary value problem, we proceed to the determination of the similarity solutions which, in turn, are used to tranform the system to a two point boundary value problem of ODEs. Finally, the reduced problem is solved numerically and its solutions are depicted for different values of the physical parameters.

2 Preliminaries

In this section, we give some preliminary notions necessary for the following analysis. Our analysis is based on the application of Lie group theory to differential equations. Any Lie group of transformation is related with the so–called infinitesimal generator, which can reproduce the finite group.

Definition 1. The infinitesimal generator of the one-parameter Lie group of transformation

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \varepsilon) \tag{2.1}$$

is the vector field

$$\mathbf{V} = \mathbf{V}(\mathbf{x}) = \xi(\mathbf{x}) \cdot \nabla = \sum_{i=1}^{n} \xi_i(\mathbf{x}) \frac{\partial}{\partial x_i}. \tag{2.2}$$

[2]For instance, in [17, 19] constant magnetic field is assumed, while in [20] a magnetic field depended on x is considered.

The components ξ_i provide the infinitesimal part of the transformation group, that is eq. (2.1) can be written as

$$\mathbf{x}^* = \mathbf{x} + \varepsilon\xi(\mathbf{x}) + o(\varepsilon) \tag{2.3}$$

Let $F(\mathbf{x}) = F(x_1, x_2, \ldots, x_n)$ be any differentiable function, and let the vector field $\mathbf{V}$ represents a transformation group. The differential operator $\mathbf{V}$ acts on the function F as follows

$$\mathbf{V}F(\mathbf{x}) = \xi(\mathbf{x}) \cdot \nabla F(\mathbf{x}) = \sum_{i=1}^{n} \xi_i(\mathbf{x}) \frac{\partial F(\mathbf{x})}{\partial x_i}. \tag{2.4}$$

If $\mathbf{V}F \equiv 0$, we will say that F is invariant under the action of the associated transformation group.
We are mainly interested in the notion of invariance of a PDE under the action of some group of transformation. Thus, the space over which the transformation group will act is made up by the dependent variables $\mathbf{u}$ and independent ones $\mathbf{x}$ involved in the differential equation. In that case, a typical transformation group has the form

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}, \mathbf{u}; \varepsilon), \tag{2.5}$$
$$\mathbf{u}^* = \mathbf{U}(\mathbf{x}, \mathbf{u}; \varepsilon). \tag{2.6}$$

The corresponding infinitesimal generator is denoted by

$$\mathbf{V} = \xi_i(x_j, u^\mu)\frac{\partial}{\partial x_i} + \eta^l(x_j, u^\mu)\frac{\partial}{\partial u^l}, \quad i, j = 1, \ldots, n, \quad l, \mu = 1, \ldots, m, \tag{2.7}$$

where n and m are the number of independent and dependent variables, respectively.

Consider now a *kth* order partial differential equation defined on a domain $\Omega_{\mathbf{x}}$ in $\mathbf{x}$-space

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, ..., \mathbf{u}^{[k]}) = 0, \tag{2.8}$$

where $\mathbf{u}^{[l]}$ denotes any l-order partial derivative of $\mathbf{u}$ with respect to $\mathbf{x}$. Noting that the differential operator Δ admits an enlarged argument containing in addition the partial derivatives of $\mathbf{u}$, we must also enlarge the infinitesimal generator in the following way [1, 2]

$$\begin{aligned}\mathbf{V} = \xi_i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x_i} + \eta^l(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^l} + \eta_i^{\mu(1)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]})\frac{\partial}{\partial u_i^\mu} \\ + \cdots + \eta_{i_1 i_2 \ldots i_k}^{\mu(k)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, \ldots, \mathbf{u}^{[k]})\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}^\mu},\end{aligned} \tag{2.9}$$

where by $u_{i_1 i_2 \ldots i_k}^\mu$ is denoted the mixed partial derivative of u^μ with respect to $x_{i_1}, \ldots, x_{i_k}$ and $\eta_{i_1 i_2 \ldots i_k}^{\mu(k)}$ denotes the additional components of the infinitesimal generator given by the recursion formula

$$\begin{aligned}\eta_{i_1 \ldots i_l}^{\mu(l)} = D_{i_l}\eta_{i_1 \ldots i_{l-1}}^{\mu(l-1)} - (D_{i_l}\xi_j)u_{i_1 \ldots i_{l-1} j}^\mu, \\ 1 \leq l \leq k, \ 1 \leq i_l \leq n, \ 1 \leq \mu \leq m.\end{aligned} \tag{2.10}$$

Let us now associate with the PDE (2.8) the boundary conditions

$$B_\alpha(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, ..., \mathbf{u}^{[k-1]}) = 0 \tag{2.11}$$

prescribed on the boundary surfaces

$$\omega_\alpha(\mathbf{x}) = 0, \quad a = 1, 2, ..., s. \tag{2.12}$$

We are ready to give the main definition concerning the invariance of a boundary value problem under the action of a transformation group.

Definition 2. The one-parameter Lie group of transformation given by eqs. (2.5)–(2.6) constitutes a symmetry group for the boundary value problem described by eqs. (2.8) and (2.11)–(2.12) if and only if

$$(i)\ \mathbf{V}^{(k)}\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, ..., \mathbf{u}^{[k]}) = 0, \ \text{ when } \ \Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, ..., \mathbf{u}^{[k]}) = 0, \tag{2.13}$$

$$(ii)\ \mathbf{V}\omega_\alpha(\mathbf{x}) = 0 \ \text{ when } \ \omega_\alpha(\mathbf{x}) = 0, \tag{2.14}$$

$$(iii)\ \mathbf{V}^{(k-1)}B_\alpha(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, ..., \mathbf{u}^{[k-1]}) = 0,$$
$$\text{when } \ B_\alpha(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, ..., \mathbf{u}^{[k-1]}) = 0 \ \text{ on } \ \omega_\alpha(\mathbf{x}) = 0. \tag{2.15}$$

3 The mathematical description of the problem

We consider a free convective, laminar boundary-layer flow of an electrically conducting incompressible viscous fluid over a vertical porous and elastic surface. The surface is stretched vertically upward along the positive x-axis, with a prescribed velocity

$$u(x, y = 0) = u_0(x), \tag{3.1}$$

while the origin $(x, y) = (0, 0)$ is kept fixed. The y–axis is vertical to the surface, as it is depicted[3] in Fig. 1. Also, due to the fact that the elastic surface is porous, there is a component of the velocity of the fluid which has vertical direction to the surface given by

$$v(x, y = 0) = v_0(x). \tag{3.2}$$

The motion of the surface within the fluid creates a boundary layer, which is extended along the x-axis. All the system is under the influence of a magnetic field $B = B(x, y)$, which applies to the y–direction. We consider that the temperature of the surface changes along the x-axis and its distribution is described by a given function $T_0(x)$.
Under the assumption that the viscous dissipation term in the energy equation and the induced magnetic field can be neglected, the basic boundary layer equations of

[3] Although the problem is set in the two dimensional space of x and y, the expression "the surface $y = 0$" is used instead of the more accurate "the curve $y = 0$".

the mass, momentum and energy for the steady flow of Boussinesq type are respectively as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.3}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma u B^2}{\rho}+g\beta(T-T_\infty), \tag{3.4}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}, \tag{3.5}$$

where σ is the electric conductivity, β is the volumetric coefficient of thermal expansion, ν is the kinematic viscosity, ρ is the mass density and a is the thermal diffusivity, which are assumed to be constants. Also, g is the gravity field assumed to be parallel to the $x-$axis, $T=T(x,y)$ is the temperature field and T_∞ is the temperature at infinity. According to the above description, the boundary conditions of the problem should be of the form

$$y=0: \quad \begin{aligned} &u(x,0)=u_0(x),\\ &v(x,0)=v_0(x), \quad x\succ 0\\ &\theta(x,0)=\theta_0(x). \end{aligned} \tag{3.6}$$

$$y\to\infty: \quad u\to 0, \quad \theta\to 0, \quad x\succ 0, \tag{3.7}$$

where $\theta=T-T_\infty$. Also, $\theta_0=T_0-T_\infty$ is a prescribed function along the boundary surface $y=0$.
We introduce now the stream function ψ, which is related to the components of the velocity field by the equations

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}. \tag{3.8}$$

Inserting eqs. (3.8) into the field equations (3.3)–(3.5), we obtain

$$\psi_y\psi_{xy}-\psi_x\psi_{yy}-\nu\psi_{yyy}+\kappa B^2\psi_y-\phi\theta=0, \tag{3.9}$$

$$\psi_y\theta_x-\psi_x\theta_y-a\theta_{yy}=0, \tag{3.10}$$

where $\kappa=\sigma/\rho$ and $\phi=g\beta$.
The associated boundary conditions can be written as

$$y=0: \quad \begin{aligned} &\psi_y(x,0)=u_0(x)\equiv\Psi_1(x),\\ -&\psi_x(x,0)=v_0(x)\equiv\Psi_2(x), \quad x\succ 0\\ &\theta(x,0)=\theta_0(x). \end{aligned} \tag{3.11}$$

$$y\to\infty: \quad \psi_y\to 0, \quad \theta\to 0, \quad x\succ 0. \tag{3.12}$$

4 Determination of the symmetry groups

In this section, we will look for any possible symmetry group of the boundary value problem described by PDEs (3.9)–(3.10) and boundary conditions (3.11)–(3.12).

The infinitesimal generator associated with the possible symmetries for the system under study has the general form[1, 2]:

$$\mathbf{V} = \xi_1(x,y,\psi,\theta)\frac{\partial}{\partial x} + \xi_2(x,y,\psi,\theta)\frac{\partial}{\partial y} + \eta^1(x,y,\psi,\theta)\frac{\partial}{\partial \psi} + \eta^2(x,y,\psi,\theta)\frac{\partial}{\partial \theta}. \tag{4.1}$$

Due to the order of the PDEs of our system, we need the third extension of (4.1). This is given by

$$\begin{aligned}\mathbf{V}^{(3)} = \mathbf{V} &+ \eta_1^{1(1)}\frac{\partial}{\partial \psi_x} + \eta_2^{1(1)}\frac{\partial}{\partial \psi_y} + \eta_1^{2(1)}\frac{\partial}{\partial \theta_x} + \eta_2^{2(1)}\frac{\partial}{\partial \theta_y} + \eta_{11}^{1(2)}\frac{\partial}{\partial \psi_{xx}} \\ &+ \eta_{12}^{1(2)}\frac{\partial}{\partial \psi_{xy}} + \eta_{22}^{1(2)}\frac{\partial}{\partial \psi_{yy}} + \eta_{11}^{2(2)}\frac{\partial}{\partial \theta_{xx}} + \eta_{12}^{2(2)}\frac{\partial}{\partial \theta_{xy}} + \eta_{22}^{2(2)}\frac{\partial}{\partial \theta_{yy}} \\ &+ \eta_{111}^{1(3)}\frac{\partial}{\partial \psi_{xxx}} + \eta_{112}^{1(3)}\frac{\partial}{\partial \psi_{xxy}} + \eta_{122}^{1(3)}\frac{\partial}{\partial \psi_{xyy}} + \eta_{222}^{1(3)}\frac{\partial}{\partial \psi_{yyy}} + \\ &\eta_{111}^{2(3)}\frac{\partial}{\partial \theta_{xxx}} + \eta_{112}^{2(3)}\frac{\partial}{\partial \theta_{xxy}} + \eta_{122}^{2(3)}\frac{\partial}{\partial \theta_{xyy}} + \eta_{222}^{2(3)}\frac{\partial}{\partial \theta_{yyy}},\end{aligned} \tag{4.2}$$

where the components $\eta^{\gamma(1)}$, $\eta^{\gamma(2)}$ and $\eta^{\gamma(3)}$ $(\gamma = 1,2)$ depend on the quantities $(x,y,\psi,\theta,\psi^{[1]},\theta^{[1]})$, $(x,y,\psi,\theta,\ \psi^{[1]},\psi^{[2]},\theta^{[1]},\theta^{[2]})$ and $(x,y,\psi,\theta,\ \psi^{[1]},\psi^{[2]},\psi^{[3]},$ $\theta^{[1]},\theta^{[2]},\theta^{[3]})$, respectively. The general form of any component of the extended infinitesimal generator is given by the following relation [2]:

$$\eta^{\mu(l)}_{i_1\ldots i_l} = D_{i_l}\eta^{\mu(l-1)}_{i_1\ldots i_{l-1}} - (D_{i_l}\xi_j)u^{\mu}_{i_1\ldots i_{l-1}j}, \quad 1 \le l \le 3,\ 1 \le i_l, \mu \le 2. \tag{4.3}$$

In the last formula, we used the denotation $u^1 := \psi$ and $u^2 := \theta$. Also, we must note that $u^{[l]}$ denotes any l-order partial derivative of u with respect to x and y. It is important to note that it is not necessary to compute all the components which appear in eq. (4.2). Having in mind the form of the PDEs of our system, we easily conclude that the components which will survive after the application of the infinitesimal generator on the system will be:

$$\eta_1^{1(1)},\ \eta_2^{1(1)},\ \eta_1^{2(1)},\ \eta_2^{2(1)},\ \eta_{12}^{1(2)},\ \eta_{22}^{2(2)},\ \eta_{22}^{1(2)},\ \eta_{222}^{1(3)}.$$

Hence, in order to develop the prescribed method, we must first calculate these components. One can easily carry out such calculation using the formula (4.3).

Let us proceed further by applying the invariance requirement on the system given by eqs. (3.9) –(3.10). The condition (2.13) will take the form:

$$\mathbf{V}^{(3)}(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy} + kB^2\psi_y - \phi\theta) = 0, \tag{4.4}$$
when eqs. (3.9)–(3.10) hold

and

$$\mathbf{V}^{(3)}(\psi_y\theta_x - \psi_x\theta_y - a\theta_{yy}) = 0, \tag{4.5}$$
when eqs. (3.9)–(3.10) hold.

Developing eqs. (4.4)–(4.5), one can straightforwardly take

$$2\xi_1\kappa BB_x\psi_y + 2\xi_2\kappa BB_y\psi_y - \phi\eta^2 - \eta_1^{1(1)}\psi_{yy} + \eta_2^{1(1)}\psi_{xy} + \kappa B^2\eta_2^{1(1)} + \\ +\eta_{12}^{1(2)}\psi_y - \eta_{22}^{1(2)}\psi_x - \nu\eta_{222}^{1(3)} = 0, \tag{4.6}$$

when the equations (3.9), (3.10)) hold

and

$$-\eta_1^{1(1)}\theta_y + \eta_2^{1(1)}\theta_x + \eta_1^{2(1)}\psi_y - \eta_2^{2(1)}\psi_x - a\eta_{22}^{2(2)} = 0, \tag{4.7}$$

when the equations (3.9), (3.10) hold.

Let us now explore the conditions given by eqs. (4.6) and (4.7). Inserting eqs. (3.9) and (3.10) into both eqs. (4.6) and (4.7) as well as the expressions of all η's given by eq. (4.3), we will obtain two identically zero polynomials in terms of various combinations of the derivatives of ψ and θ. Thus, their coefficients, which will contain ξ_1, ξ_2, η^1 and η^2 and their derivatives, should be zero. This procedure provides the so–called determining equations, i.e., PDEs in ξ_1, ξ_2, η^1 and η^2, the solutions of which determine the components of the infinitesimal generator. In our problem, the determining equations associated with the conditions (4.6) and (4.7) are given by

$$\xi_{1,\theta} = 0, \quad \xi_{1,y} = 0, \quad \xi_{1,\psi} = 0, \quad \xi_{2,\theta} = 0, \quad \xi_{2,\psi} = 0, \tag{4.8}$$

$$\xi_{2,yy} = 0, \quad \eta_x^1 = 0, \quad \eta_y^1 = 0, \quad -\phi\eta^2 + \eta_\psi^1\phi\theta - 3\phi\theta\xi_{2,y} = 0, \tag{4.9}$$

$$\eta_\theta^1 = 0, \quad \eta_{\psi\psi}^1 = 0, \quad \xi_{2,xy} = 0, \quad \eta_\psi^1 - \xi_{1,x} + \xi_{2,y} = 0, \tag{4.10}$$

$$2\kappa B(x,y)B_x(x,y)\xi_1 + 2\kappa B(x,y)B_y(x,y)\xi_2 + 2\kappa B^2(x,y)\xi_{2,y} = 0, \tag{4.11}$$

$$\eta_y^2 = 0, \quad \eta_x^2 = 0, \quad \eta_\psi^2 = 0, \quad \eta_{\theta\theta}^2 = 0. \tag{4.12}$$

One can confirm[4] that the solutions of the system (4.8)–(4.12) are given as follows:

$$\begin{aligned} \xi_1 &= \kappa_1 x + \kappa_1', \\ \xi_2 &= A(x) + sy + \kappa_2', \\ \eta^1 &= (\kappa_1 - s)\psi + \kappa_3', \\ \eta^2 &= (\kappa_1 - 4s)\theta, \end{aligned} \tag{4.13}$$

where $\kappa_1, s, \kappa_1', \kappa_3'$ are arbitrary constants and A is an arbitrary function.
Furthermore, we have to apply the invariance condition to the boundary conditions [1, 2]; that is, on the relation which should be held along the boundary surfaces and on the boundary surfaces themselves. We start with the latter requirement; We can write the two boundary surfaces of the problem under study as follows

$$\omega_1(x,y) = y, \qquad \omega_2(x,y) = y - K, \quad K \in I\!R^+.$$

Thus, the invariance requirement for the former takes the form

$$\mathbf{V}[\omega_1] = 0 \ \text{ when } \ \omega_1 = 0 \Rightarrow \xi_2(x,0) = 0. \tag{4.14}$$

[4] We remark that the system (4.8)–(4.12) is overdetermined, so one can obtain its solution by simple integrations.

The latter fulfils the invariance requirement identically, thus, it does not impose any constraint on the generator.
Inserting eq. (4.14) into eqs. (4.13), we obtain the final form for the components of the infinitesimal generator

$$\begin{aligned} \xi_1 &= \kappa_1 x + \kappa_1', \\ \xi_2 &= sy, \\ \eta^1 &= (\kappa_1 - s)\psi + \kappa_3', \\ \eta^2 &= (\kappa_1 - 4s)\theta. \end{aligned} \tag{4.15}$$

What it remains is to require invariance of the data which must be held on the boundary surfaces. This requirement means

$$\begin{aligned} &\mathbf{V}^{(1)}[\psi_y - \Psi_1(x)] = 0 \quad \text{when} \quad \psi_y(x,0) = \Psi_1(x), \\ &\mathbf{V}^{(1)}[-\psi_x - \Psi_2(x)] = 0 \quad \text{when} \quad \psi_x(x,0) = -\Psi_2(x), \\ &\mathbf{V}^{(1)}[\theta - \theta_0(x)] = 0 \quad \text{when} \quad \theta(x,0) = \theta_0(x). \end{aligned} \tag{4.16}$$

Examining the above conditions, we obtain the following differential equations

$$\begin{aligned} &(\kappa_1 x + \kappa_1')\Psi_1' - (\kappa_1 - 2s)\Psi_1 = 0, \\ &(\kappa_1 x + \kappa_1')\Psi_2' + s\Psi_2 = 0, \\ &(\kappa_1 x + \kappa_1')\theta_0' - (\kappa_1 - 4s)\theta_0 = 0, \end{aligned}$$

which directly give the admissible form for the functions Ψ_1, Ψ_2 and θ_0

$$\begin{aligned} \Psi_1(x) &= c_1|\kappa_1 x + \kappa_1'|^{\frac{\kappa_1 - 2s}{\kappa_1}} \\ \Psi_2(x) &= c_2|\kappa_1 x + \kappa_1'|^{-\frac{s}{\kappa_1}}, \\ \theta_0(x) &= c_3|\kappa_1 x + \kappa_1'|^{\frac{\kappa_1 - 4s}{\kappa_1}}. \end{aligned} \tag{4.17}$$

Consequently, a set of boundary conditions conformed to the symmetries (4.15) should be of the form

$$\begin{aligned} y = 0: \qquad & \psi_y(x,0) = c_1|\kappa_1 x + \kappa_1'|^{\frac{\kappa_1 - 2s}{\kappa_1}}, \\ & \psi_x(x,0) = -c_2|\kappa_1 x + \kappa_1'|^{-\frac{s}{\kappa_1}}, \quad x \succ 0 \\ & \theta(x,0) = c_3|\kappa_1 x + \kappa_1'|^{\frac{\kappa_1 - 4s}{\kappa_1}}. \end{aligned} \tag{4.18}$$

and

$$y \to \infty: \qquad \psi_y \to 0, \quad \theta \to 0, \quad x \succ 0. \tag{4.19}$$

It is interesting to note that the system of determining equations provides a constraint for the magnetic field function, as well. Indeed, eq. (4.11) takes the form

$$(\kappa_1 x + \kappa_1')B_x + syB_y + sB = 0.$$

Hence, B is constrained to be of the form

$$B(x,y) = \frac{1}{y} G(\frac{|\kappa_1 x + \kappa_1'|^s}{y^{\kappa_1}}), \tag{4.20}$$

where G is an arbitrary function.

Concluding, the following statement has been proved.

Proposition. *The boundary value problem described by eqs.* (3.9)–(3.10) *and the data* (3.11)–(3.12), *admits the following multi-parameter group of symmetries*

$$\begin{aligned} x^* &= x + \varepsilon(\kappa_1 x + \kappa_1'), \\ y^* &= y + \varepsilon s y, \\ \psi^* &= \psi + \varepsilon((\kappa_1 - s)\psi + \kappa_3'), \\ \theta^* &= \theta + \varepsilon((\kappa_1 - 4s)\theta). \end{aligned} \tag{4.21}$$

Moreover, the admissible form of the data[5] *on the boundaries should be of the form given by eq.* (4.17) *and the magnetic field function is constrained to be of the form given by eq.* (4.20).

Looking at the transformation equations (4.21), one can recognize two kind of symmetries. Vanishing the parameters κ_1' and κ_3', the scaling group parameterized by κ_1 and s arises. On the other hand, vanishing κ_1 and s, the group of translations with respect to x and ψ is obtained.

5 Group–invariant solutions

The next question is whether the symmetry group we have obtained in the last section gives any of the so–called group–invariant solutions. A group–invariant solution is nothing else but a solution of the BVP (3.9)–(3.12), which is also invariant under the group (4.21) [1, 2]. Suppose (ψ, θ) is a solution of the problem (3.9)–(3.12). In order this solution to be invariant under the transformation group (4.21), the following system of partial differential equations must hold

$$\xi_1 \frac{\partial \psi}{\partial x} + \xi_2 \frac{\partial \psi}{\partial y} = \eta^1, \quad \xi_1 \frac{\partial \theta}{\partial x} + \xi_2 \frac{\partial \theta}{\partial y} = \eta^2$$

or

$$-(\kappa_1 x + \kappa_1')\frac{\partial \psi}{\partial x} - sy\frac{\partial \psi}{\partial y} + (\kappa_1 - s)\psi = -\kappa_3', \tag{5.1}$$

$$-(\kappa_1 x + \kappa_1')\frac{\partial \theta}{\partial x} - sy\frac{\partial \theta}{\partial y} + (\kappa_1 - 4s)\theta = 0. \tag{5.2}$$

[5] In the sense that they respect the obtained symmetries and so they do not destroy the symmetry of the boundary value problem

Using the method of characteristics, we can solve the system (5.1)–(5.2)

$$\psi(x,y) = |\kappa_1 x + \kappa_1^{'}|^{\frac{\kappa_1 - s}{\kappa_1}} F(X_1) - \frac{k_3^{'}}{k_3}, \tag{5.3}$$

$$\theta(x,y) = |\kappa_1 x + \kappa_1^{'}|^{\frac{\kappa_1 - 4s}{\kappa_1}} H(X_1), \tag{5.4}$$

where F, H are arbitrary functions and X_1 is the similarity variable given by the relation

$$X_1 = \frac{|\kappa_1 x + \kappa_1^{'}|^s}{y^{\kappa_1}}. \tag{5.5}$$

Equations (5.3)–(5.4) give the general form for any group–invariant solution of our problem. The interesting point here is that such a solution has the property to reduce the number of the independent variables of the problem. Thus, inserting the solution (5.3)–(5.4) and the admissible form of the magnetic field (4.20) into the field equations (3.9)–(3.10), we obtain

$$-\kappa_1^2(\kappa_1 - 2s) X_1^{\frac{2\kappa_1+2}{\kappa_1}} F^{'2} - \kappa_1(\kappa_1 - s)(\kappa_1+1) X_1^{\frac{\kappa_1+2}{\kappa_1}} F F^{'} + \nu \left[\kappa_1(\kappa_1+1)(\kappa_1+2) X_1^{\frac{\kappa_1+3}{\kappa_1}} F^{'} + \kappa_1^2 (3\kappa_1+3) X_1^{\frac{2\kappa_1+3}{\kappa_1}} F^{''} + \kappa_1^3 X_1^{\frac{3\kappa_1+3}{\kappa_1}} F^{'''} \right] - \kappa_1 \kappa G^2 X_1^{\frac{\kappa_1+3}{\kappa_1}} F^{'} + \kappa_1^2(\kappa_1 - s) X_1^{\frac{2\kappa_1+2}{\kappa_1}} F F^{''} - \phi H = 0, \tag{5.6}$$

$$-(\kappa_1 - 4s) X_1^{\frac{\kappa_1+1}{\kappa_1}} F^{'} H - \kappa_1(\kappa_1 - s) X_1^{\frac{\kappa_1+1}{\kappa_1}} F H^{'} - \alpha \left[\kappa_1(\kappa_1+1) X_1^{\frac{\kappa_1+2}{\kappa_1}} H^{'} + \kappa_1^2 X_1^{\frac{2\kappa_1+2}{\kappa_1}} H^{''} \right] = 0, \tag{5.7}$$

Also, inserting eqs. (5.3)–(5.4) into the boundary conditions (4.18)–(4.19), we take the following boundary conditions

$$F(0) = -\frac{c_2}{\kappa_1 - s}, \quad F^{'}(0) = -\frac{c_1}{\kappa_1} \lim_{X_1 \to 0+} \left[X_1^{-\frac{\kappa_1+1}{\kappa_1}} \right], \quad H(0) = c_3, \tag{5.8}$$

$$\lim_{X_1 \to \infty} F^{'}(X_1) = 0, \quad \lim_{X_1 \to \infty} H(X_1) = 0, \tag{5.9}$$

where $\kappa_1 \in [-1, 0)$ and $\kappa_1 \neq s$.
Eqs. (5.6)–(5.9) describe the new form of our problem. Thus, the initial boundary value problem of PDEs has been transformed to a boundary value problem of ODEs which is generally easier to be solved by some numerical method.

6 Numerical results for the scaling symmetry

Proceeding further to numerical results, we are confined to the case of scaling symmetry, consequently we choose $\kappa_1^{'} = \kappa_3^{'} = 0$. Furthermore, we examine two distinct cases. The first one for the case $\kappa_1 = -1$, corresponding to the problem already studied by [20] and the second one for the case $\kappa_1 = -1/2$, which corresponds to a magnetic field depended on both variables x and y.

Case 1 $\kappa_1 = -1$
Moreover, in order to produce results comparable to [20], we set $s = \frac{m-1}{2}$, where m is an arbitrary parameter. Thus, we examine the case in which the transformation equations are of the following form:

$$\begin{aligned} x^* &= e^{-\varepsilon} x, \\ y^* &= e^{\varepsilon s} y, \\ \psi^* &= e^{-(1+s)\varepsilon} \psi, \\ \theta^* &= e^{-(1+4s)\varepsilon} \theta. \end{aligned} \tag{6.1}$$

Also, under this choice of the parameters, the similarity solutions (5.3)–(5.4) take the form

$$\psi(x, y) = x^{\frac{m+1}{2}} F(X_1), \tag{6.2}$$

$$\theta(x, y) = x^{2m-1} H(X_1), \tag{6.3}$$

while the similarity variable (5.5) reduces to

$$X_1 = y x^{\frac{m-1}{2}}. \tag{6.4}$$

As far as the magnetic field is concerned, we choose the function G to be of the form

$$G(X_1) = C X_1,$$

where C an arbitrary non–vanishing constant. The above equation with the aid of eq. (4.20) leads to a magnetic field depended only on x

$$B(x, y) = B_0 x^{\frac{m-1}{2}}, \tag{6.5}$$

where B_0 is a constant.
Also, the boundary conditions associated with this choice of parameters become

$$\begin{aligned} y = 0: \quad & \psi_y(x, 0) = c_1 x^m, \\ & \psi_x(x, 0) = -c_2 x^{\frac{m-1}{2}}, \quad x \succ 0 \\ & \theta(x, 0) = c_3 x^{2m-1} \end{aligned} \tag{6.6}$$

$$y \to \infty: \quad \psi_y \to 0, \quad \theta \to 0, \quad x \succ 0. \tag{6.7}$$

Finally, the reduced system of ODEs corresponding to the above problem is given by the equations:

$$\nu F''' + \frac{m+1}{2} F F'' - m F'^2 - M F' + \phi H = 0, \tag{6.8}$$

$$\frac{1}{Pr} H'' + \left(\frac{m+1}{2\nu}\right) F H' - \left(\frac{2m-1}{\nu}\right) F' H = 0, \tag{6.9}$$

where $M = \kappa B_0^2 = \sigma B_0^2/\rho$.
Also, the associated boundary conditions take the form

$$F(0) = -\frac{2c_2}{m+1} = F_m, \quad F'(0) = c_1, \quad H(0) = c_3, \tag{6.10}$$

$$\lim_{X_1\to\infty} F'(X_1) = 0, \quad \lim_{X_1\to\infty} H(X_1) = 0. \tag{6.11}$$

Eqs. (6.8)–(6.11) arise either from the initial problem (3.9)–(3.12) by the insertion of eqs. (6.2)–(6.5) accompanied by boundary conditions (6.6)–(6.7) or directly from the transformed problem given by eqs. (5.6)–(5.9), substituting the chosen parameters.

In order to face numerically the problem (6.8)–(6.11), we have used a numerical solver of MATLAB package which solves any two–point boundary value problem for ODEs by collocation. We have obtained results for the fields $F'(X_1)$ and $H(X_1)$ corresponding to the velocity and temperature fields, respectively. In Fig. 2, it is shown the behaviour of velocity field versus X_1 for different values of the parameter m. We remark that the velocity profile decreases as the value of m increases. In Fig. 3, it is examined the behaviour of the velocity for different choices of the boundary data at $X_1 = 0$. The influence of the kinematic viscosity on the velocity is shown in Fig. 4. As one expects, the velocity increases as the viscosity increases. In Fig. 5, it is presented the behaviour of velocity field for various values of M, which is related with the magnetic field intensity. One can see that as the magnetic field intensity grows up the velocity field decreases. The temperature field follows a similar to the velocity field behaviour (see Fig 6, 7, 8), except the case of correlation with the magnetic field intensity. In Fig. 9, one can notice that the temperature field, unlike the velocity field, increases as the parameter M increases.

Case 2 $\kappa_1 = -\frac{1}{2}$
In this case, as the function G is concerned, we keep the same form as in the previous case, but we choose $\kappa_1 = -\frac{1}{2}$ and $s = -\frac{1}{4}$. This choice allows the function B to depend on both x and y variables. Indeed, in virtue of eq. (4.20), we easily conclude that

$$B(x) = 2^{\frac{1}{4}} B_0 x^{-\frac{1}{4}} y^{-\frac{1}{2}}.$$

The above chosen parameters provide the following group of transformations

$$\begin{aligned} x^* &= e^{-\frac{\varepsilon}{2}} x, \\ y^* &= e^{-\frac{\varepsilon}{4}} y, \\ \psi^* &= e^{-\frac{\varepsilon}{4}} \psi, \\ \theta^* &= e^{\frac{\varepsilon}{2}} \theta. \end{aligned} \tag{6.12}$$

Eqs. (5.3)–(5.4) provide the similarity solution for the case under study as follows

$$\psi(x,y) = 2^{-\frac{1}{2}} x^{\frac{1}{2}} F(X_1), \tag{6.13}$$

$$\theta(x,y) = 2x^{-1} H(X_1), \tag{6.14}$$

with similarity variable

$$X_1 = 2^{\frac{1}{4}} x^{-\frac{1}{4}} y^{\frac{1}{2}}. \tag{6.15}$$

Also, the acceptable boundary conditions for these values of parameters, become

$$\begin{aligned} y = 0: \quad & \psi_y(x,0) = c_1 \\ & \psi_x(x,0) = -2^{\frac{1}{2}} c_2 x^{-\frac{1}{2}}, \quad x \succ 0 \\ & \theta(x,0) = 2c_3 x^{-1} \end{aligned} \tag{6.16}$$

$$y \to \infty: \quad \psi_y \to 0, \quad \theta \to 0, \quad x \succ 0. \tag{6.17}$$

Finally, for the similarity solution given by eqs. (6.13)–(6.14), the system of partial differential equations (3.9)–(3.10) reduce to the following system of ODEs

$$FF' + X_1 FF'' + 6\nu X_1^{-2} F' - 6\nu X_1^{-1} F'' + 2\nu F''' - 8MF' + \phi H = 0, \tag{6.18}$$

$$4F'H + FH' - 2\alpha X_1^{-2} H' + 2\alpha X_1^{-1} H'' = 0 \tag{6.19}$$

and the boundary conditions take the form

$$F(0) = 4c_2, \quad \lim_{X_1 \to 0+} \frac{F'(X_1)}{X_1} = 2c_1, \quad H(0) = c_3, \tag{6.20}$$

$$\lim_{X_1 \to \infty} F'(X_1) = 0, \quad \lim_{X_1 \to \infty} H(X_1) = 0. \tag{6.21}$$

The boundary value problem given by eqs. (6.18)–(6.21) is solved numerically on the interval $[0.01798, 8]$, for $c_1 = 1, c_2 = \frac{5}{4}$ and $c_3 = 1$. In Figures 10 and 11, the behaviour of the fields F'/X_1 and H versus X_1 is depicted.

7 Concluding remarks

Using group–theoretical methods, we have obtained the transformation groups for the problem under study. We have found that apart from the scaling group the system admits a group of translations, as well. Concerning the group of scaling and the associated similarity solutions, our results are in full agreement with the work of [20]. Moreover, due to the generality of our procedure and the lack of unnecessary assumptions, we have obtained the general form of the functions involved in the boundary conditions (see eqs. 4.18) and the admissible form of the magnetic field function (eq. 4.20). This enlarges the range of particular problems, probably of practical interest, which can be solved with the similarity methods. Exploiting this fact we have provided a particular example where the magnetic field function, unlike the already existing results, depends on both space variables.

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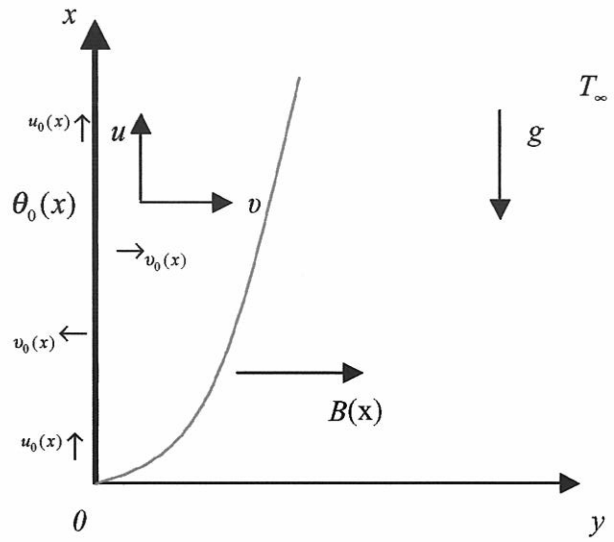
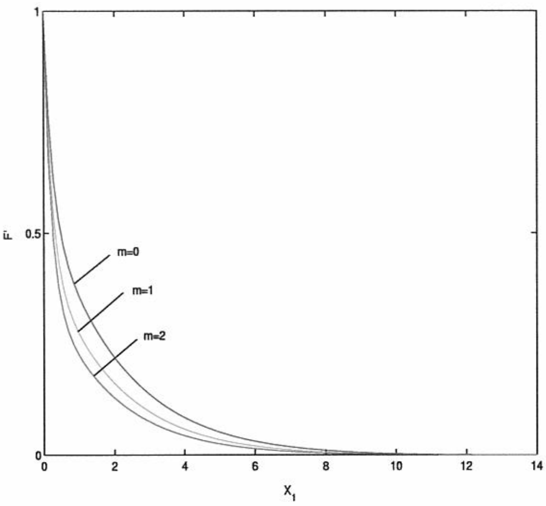
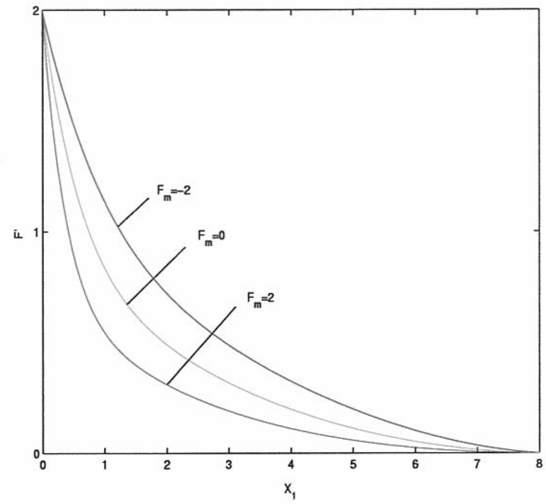
Fig. 1 Boundary-layer around a stretching surface

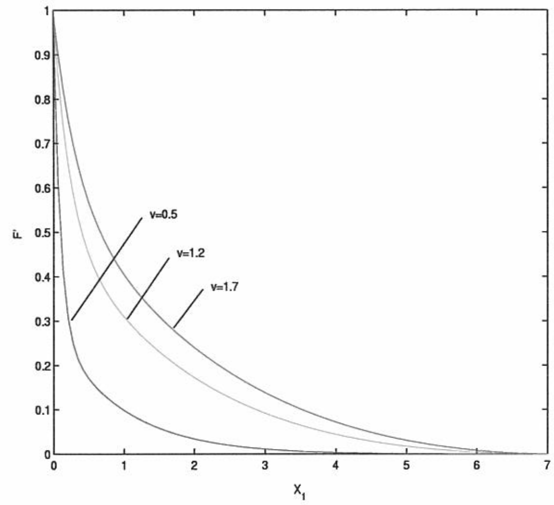




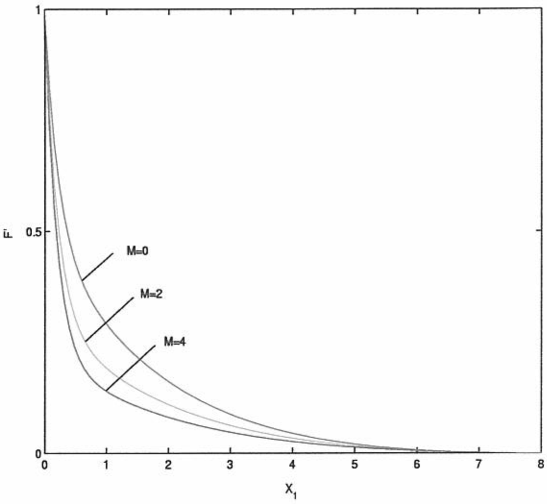
(a) Fig. 2 Behaviour of F' for different m



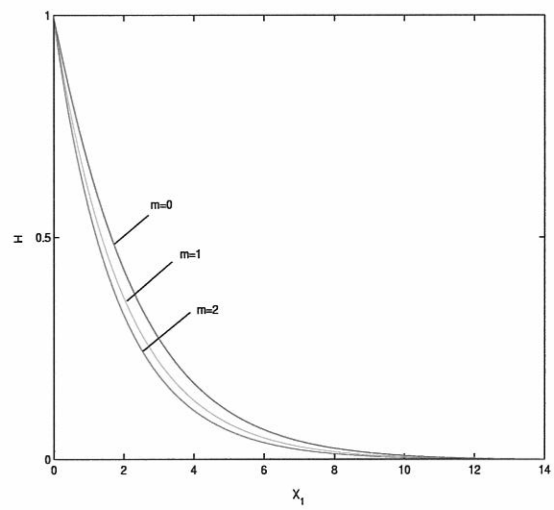
(b) Fig. 3 Behaviour of F' for different F_m



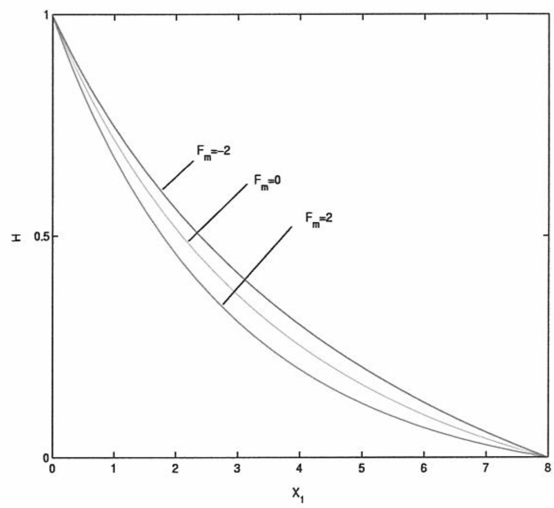
(c) Fig. 4 Behaviour of F' for different v



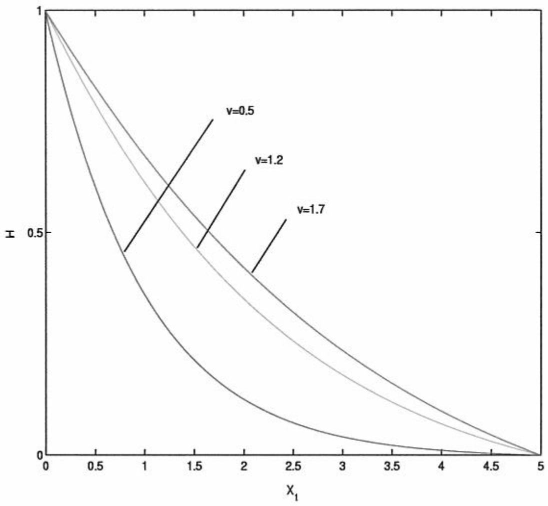
(d) Fig. 5 Behaviour of F' for different M



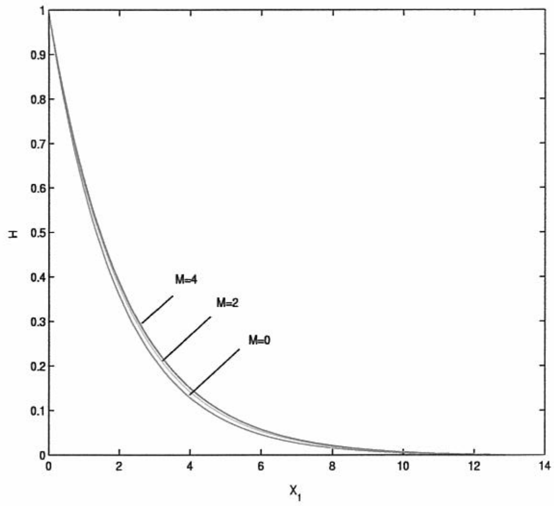
(e) Fig. 6 Behaviour of H for different m



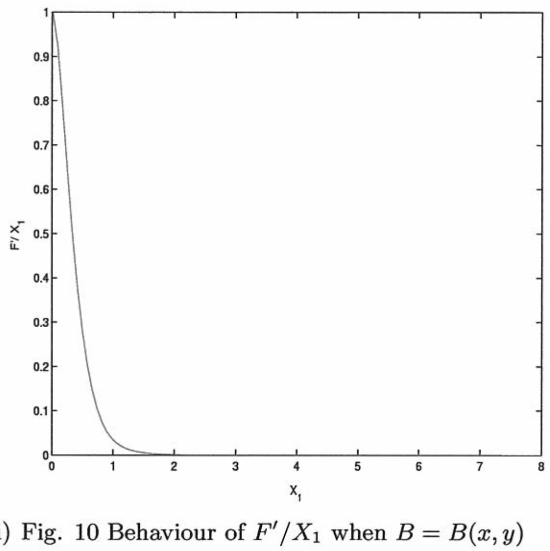
(f) Fig. 7 Behaviour of H for different F_m



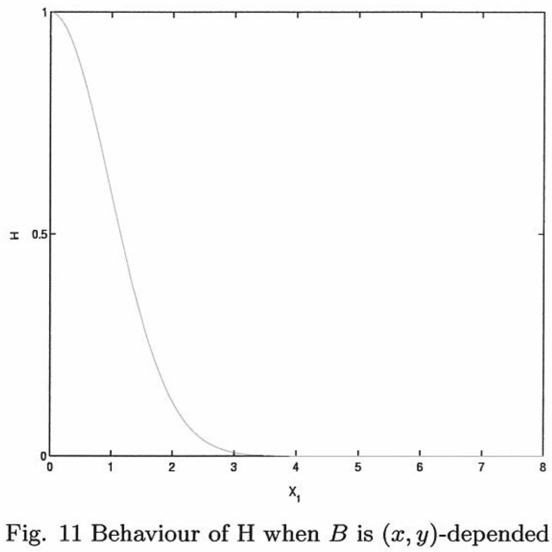
(g) Fig. 8 Behaviour of H for different v



(h) Fig. 9 Behaviour of H for different M



(i) Fig. 10 Behaviour of F'/X_1 when $B = B(x, y)$



(j) Fig. 11 Behaviour of H when B is (x, y)-depended

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